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# The Spinor Helicity Method in Dimensional Regularization

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**Abstract**

I present a rule for computing dimensionally-regularized amplitudes within the framework of the spinor-helicity method, along with explicit examples for the four- and five-gluon amplitudes.

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The helicity approach using the spinor-helicity basis [1,2, 3] has proven an extremely powerful method for computing amplitudes in gauge theories. In this approach, one computes the various helicity amplitudes for the process of interest; the squared matrix element is then given by a sum over helicities of the helicity amplitudes, each squared. The spinor-helicity basis has allowed the derivation of compact formulæ for many processes. It allows for the numerical computation of the amplitude itself (the matrix element squared is then obtained by squaring numerically); this can be a substantial advantage, since analytical formulæ for the matrix element squared are often much more complicated than those for the amplitudes.

In the usual spinor-helicity basis, one works in four dimensions, so that there are exactly two gluon helicities. In computing radiative corrections (both virtual and real), however, one often uses a dimensional regularization scheme continuing amplitudes to $4-\epsilon$ dimensions, where there are $2-\epsilon$ gluon helicities. The purpose of this Letter is to present a simple rule for computing dimensionally-regularized matrix elements within the framework of the spinor-helicity method.

In four dimensions, we have as noted above two physical degrees of freedom for the gluons, and hence two helicities. In $n+4$ dimensions, we will have an additional $n$ helicities, for which we must introduce polarization vectors. We may choose these vectors to be orthogonal to the usual four dimensions. Since all external momenta are kept in four dimensions (the justification for this will be explained later), this ensures that all the products $k\cdot\varepsilon'$ vanish, where $\varepsilon'$ are the polarization vectors for the additional helicities. The polarization vectors for the ordinary helicities, $\varepsilon^{(+)}$ and $\varepsilon^{(-)}$, will have vanishing components in the additional dimensions, and so $\varepsilon^{(\pm)}\cdot\varepsilon'$ also vanishes. This leaves us with the question of determining $\varepsilon'_i\cdot\varepsilon'_j$ for the additional helicities.

The spinor helicity basis form for $\varepsilon^{(\pm)}$,

$$\varepsilon^{(+)}_\mu(k;q)=\frac{\langle q-|\,\gamma_\mu\,|k-\rangle}{\sqrt{2}\,\langle q\,k\rangle},\qquad \varepsilon^{(-)}_\mu(k;q)=\frac{\langle q+|\,\gamma_\mu\,|k+\rangle}{\sqrt{2}\,[k\,q]} \tag{1}$$

is given in terms of massless chiral spinors $|k\pm\rangle$ carrying four-momentum $k$, a reference four-momentum $q$ (satisfying $q^2=0$ and $q\cdot k\neq 0$), and the spinor products

$$\langle q\,k\rangle=\langle q-|\,k+\rangle\,,\qquad [q\,k]=\langle q+|\,k-\rangle \tag{2}$$

which have the explicit forms [2]

$$\begin{aligned}\langle k_1\,k_2\rangle &= \sqrt{(k_1^t-k_1^z)\,(k_2^t+k_2^z)}\exp(i\,\mathrm{atan}(k_1^y/k_1^x))-(1\leftrightarrow 2)\\ &=\sqrt{\frac{k_2^t+k_2^z}{k_1^t+k_1^z}}\,(k_1^x+ik_1^y)-(1\leftrightarrow 2)\\ [k_1\,k_2] &= \mathrm{sign}(k_1^tk_2^t)\,(\langle k_2\,k_1\rangle)^*\,.\end{aligned} \tag{3}$$

It is also convenient to define a notation for the Lorentz product,

$$(k_1\, k_2) = 2k_1 \cdot k_2 = \langle k_1\, k_2\rangle\, [k_2\, k_1] \tag{4}$$

In the remainder of the paper, I shall use abbreviations of the form $\langle 1\, 2\rangle = \langle k_1\, k_2\rangle$. Using Fierz identities, we obtain the following results for dot products of the polarization vectors,

$$\begin{aligned}
\varepsilon^{(+)}(k;q)\cdot\varepsilon^{(+)}(k';q) &= 0 = \varepsilon^{(-)}(k;q)\cdot\varepsilon^{(-)}(k';q)\\
\varepsilon^{(+)}(k;q)\cdot\varepsilon^{(-)}(k';q) &= \frac{\langle q\, k'\rangle\,[q\, k]}{\langle q\, k\rangle\,[k\, q]}
\end{aligned} \tag{5}$$

The rules for extending the basis to $4+\epsilon$ dimensions take the last equation (up to a phase) as the definition of the dot product of two polarization vectors representing additional ('$[\epsilon]$') helicities:

$$\varepsilon^{([\epsilon])}(k;q)\cdot\varepsilon^{([\epsilon])}(k';q) = -\delta^{i_1 i_2}_{(\epsilon)} \tag{6}$$

(the phase is chosen so that the expression yields a value identical to that of equation (5) if $k' = k$). In this expression, $i_1$ and $i_2$ run over the $\epsilon$ additional dimensions; thus, $\mathrm{Tr}\,\delta_{(\epsilon)} = \epsilon$. [In $4-\epsilon$ dimensions, we would have $\mathrm{Tr}\,\delta_{(-\epsilon)} = -\epsilon$.] In squaring the amplitude, one must contract these indices. One also has the equations discussed earlier,

$$\begin{aligned}
k\cdot\varepsilon^{([\epsilon])}(k';q) &= 0\\
\varepsilon^{(\pm)}(k;q)\cdot\varepsilon^{([\epsilon])}(k';q') &= 0
\end{aligned} \tag{7}$$

Let us consider first the four-point amplitude. One can write [4,5,6] the on-shell tree-level amplitude for $n$-gluon scattering as a sum over non-cyclic permutations of the external legs,

$$\mathcal{A}_n\left(\{k_i,\varepsilon_i,a_i\}\right) = g^{n-2}\sum_{\sigma\in S_n/Z_n}\mathrm{Tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}\right)\, A_n\left(k_{\sigma(1)},\varepsilon_{\sigma(1)};\ldots;k_{\sigma(n)},\varepsilon_{\sigma(n)}\right) \tag{8}$$

where $k_i$, $\varepsilon_i$, and $a_i$ are respectively the momentum, polarization vector, and color index of the $i$-th external gluon. The $T^a$ are the set of hermitian traceless $N\times N$ matrices (normalized so that $\mathrm{Tr}(T^aT^b) = \delta^{ab}$), and $S_n/Z_n$ is the set of non-cyclic permutations of $\{1,\ldots,n\}$.

The *partial amplitudes* $A_j$ possess a number of nice properties. Each is gauge invariant, that is invariant under the substitution $\varepsilon_i \to \varepsilon_i + \lambda k_i$ for each leg independently. It is also invariant under cyclic permutation of its arguments, and satisfies a reflection identity,

$$A_n(n,\ldots,1) = (-1)^n A_n(1,\ldots,n) \tag{9}$$

as well as a "twist" or decoupling identity,

$$\sum_{\sigma\in Z_{n-1}} A_n(\sigma_1,\ldots,\sigma_{n-1},n) = 0. \tag{10}$$

For the four- and five-point amplitudes, this leads to a simple form for the amplitude squared summed over all outgoing colors and averaged over incoming colors,

$$
\begin{aligned}
|\mathcal{A}_4|^2 &= g^4 \frac{N^2}{N^2-1} \sum_{\sigma\in S_4/Z_4} |A_4(\sigma)|^2 \\
|\mathcal{A}_5|^2 &= g^6 \frac{N^3}{N^2-1} \sum_{\sigma\in S_5/Z_5} |A_5(\sigma)|^2
\end{aligned}
\tag{11}
$$

This equation holds for each helicity amplitude independently.

Although it is possible to formulate a recurrence relation [7,8] for the $\epsilon$-helicity amplitudes, here I shall simply calculate from the explicit form of the partial amplitudes as formal polynomials in the momenta and polarization vectors; the four-point partial amplitude may be readily extracted from the four-vector amplitude in string theory [9], while the five-point amplitude has been calculated by Lee, Nair, and the author [5]. Using the rules introduced above, we find for $n = 4$ or 5 in $4-\epsilon$ dimensions,

$$
\begin{aligned}
A_n([\epsilon]_{j_1} + \cdots + [\epsilon]_{j_2} + \cdots + ) &= 0 \\
A_n([\epsilon]_{j_1} + \cdots + [\epsilon]_{j_2} + \cdots -_m + \cdots + ) &= -i\delta^{j_1 j_2}_{(-\epsilon)} \frac{\langle j_1\, m\rangle^2 \langle j_2\, m\rangle^2}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1\, n\rangle\langle n\,1\rangle} \\
A_n([\epsilon]_{j_1} + \cdots + [\epsilon]_{j_2} + \cdots + [\epsilon]_{j_3} + \cdots + [\epsilon]_{j_4} + \cdots + ) &= \\
i\delta^{j_1 j_2}_{(-\epsilon)}\delta^{j_3 j_4}_{(-\epsilon)} \frac{\langle 1\,3\rangle\langle 2\,4\rangle\langle 1\,4\rangle\langle 2\,3\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1\, n\rangle\langle n\,1\rangle} &- i\delta^{j_1 j_3}_{(-\epsilon)}\delta^{j_2 j_4}_{(-\epsilon)} \frac{\langle 1\,2\rangle\langle 3\,4\rangle\langle 1\,4\rangle\langle 2\,3\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1\, n\rangle\langle n\,1\rangle} \\
&+ i\delta^{j_1 j_4}_{(-\epsilon)}\delta^{j_2 j_3}_{(-\epsilon)} \frac{\langle 1\,2\rangle\langle 3\,4\rangle\langle 1\,3\rangle\langle 2\,4\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1\, n\rangle\langle n\,1\rangle} \\
A_n(\text{odd number of `}[\epsilon]\text{'}) &= 0
\end{aligned}
\tag{12}
$$

for any values and ordering of the $j_i$ and $m$. We also have the ordinary Parke-Taylor formulæ [10] for the usual helicities,

$$
\begin{aligned}
A_n( + \cdots + ) &= 0 \\
A_n( - + \cdots + ) &= 0 \\
A_n( + \cdots -_{j_1} + \cdots -_{j_2} + \cdots + ) &= i \frac{\langle j_1\, j_2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1\, n\rangle\langle n\,1\rangle}
\end{aligned}
\tag{13}
$$

Squaring these amplitudes, contracting the indices on the $\delta$ tensors, and summing over the helicities $(+,-,[\epsilon])$ yields for the four-point matrix element (ignoring color factors and not averaging over incoming helicities)

$$
\begin{aligned}
g^4 &\left(\frac{1}{s^2t^2}+\frac{1}{t^2u^2}+\frac{1}{s^2u^2}\right)\Big(4\left(s^4+t^4+u^4\right)\\
&\quad -4\epsilon\left(u^2t^2+s^2t^2+s^2u^2+s^2u^2+s^2t^2+u^2t^2\right)\\
&\quad -4\epsilon\left(stu^2+st^2u+s^2tu\right)\\
&\quad +2\epsilon^2\left(t^2u^2+s^2u^2+s^2t^2\right)\Big)\\
=&16\left(1-\frac{\epsilon}{2}\right)^2\frac{\left(s^2+st+t^2\right)^3}{s^2t^2(s+t)^2}
\end{aligned}
\tag{14}
$$

in agreement with the result of Ellis and Sexton [11] (note that these authors work in $4-2\epsilon$ rather than $4-\epsilon$ dimensions),

$$
16g^4\left(1-\frac{\epsilon}{2}\right)^2\left(3-\frac{ut}{s^2}-\frac{us}{t^2}-\frac{st}{u^2}\right)
\tag{15}
$$

For the five-point matrix element, one obtains the result

$$
\begin{aligned}
\sum_{\text{helicities}}|A_5(1,2,3,4,5)|^2 =& \frac{2}{(1\,2)(2\,3)(3\,4)(4\,5)(5\,1)}\\
&\times\Bigg(\sum_{\substack{1\le j_1\le 5\\ j_1<j_2\le 5}}(j_1\,j_2)^4-\epsilon\sum_{\substack{1\le j_1\le 5\\ j_1<j_2\le 5\\ 1\le m\le 5}}(j_1\,m)^2(j_2\,m)^2\\
&\quad -\epsilon\sum_{\sigma\in Z_5}f_1(\sigma(1),\cdots,\sigma(5))+\epsilon^2\sum_{\sigma\in Z_5}f_2(\sigma(1),\cdots,\sigma(5))\Bigg)
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
f_1(1,2,3,4,5) =& -(1\,4)(2\,3)\Big(\langle 1\,2\rangle\langle 3\,4\rangle[3\,1][4\,2]+[2\,1][4\,3]\langle 1\,3\rangle\langle 2\,4\rangle\Big)\\
&-(1\,3)(2\,4)\Big(\langle 1\,2\rangle\langle 4\,3\rangle[4\,1][3\,2]+[2\,1][3\,4]\langle 1\,4\rangle\langle 2\,3\rangle\Big)\\
&-(1\,2)(3\,4)\Big(\langle 1\,4\rangle\langle 2\,3\rangle[3\,1][4\,2]+[4\,1][3\,2]\langle 1\,3\rangle\langle 2\,4\rangle\Big)\\
=& 2\Big((1\,2)^2(3\,4)^2+(1\,4)^2(2\,3)^2+(1\,3)^2(2\,4)^2\Big)-\Big((1\,2)(3\,4)+(1\,4)(2\,3)+(1\,3)(2\,4)\Big)^2\\
f_2(1,2,3,4,5) =& (1\,2)(3\,4)(1\,4)(2\,3)+(1\,2)(3\,4)(1\,3)(2\,4)+(1\,4)(2\,3)(1\,3)(2\,4)
\end{aligned}
\tag{17}
$$

which also agrees with the result of Ellis and Sexton [11], equation (3.12), after expressing both in terms of a non-redundant set of invariants and removing color factors (note that the normalization of the ordinary product '$(i\,j)$' here differs from that used in ref. [11]).

In order to understand why the prescriptions proposed here work, let us consider squaring the amplitude and summing over helicities without resort to the spinor helicity method. Each partial amplitude is a sum of term, each of which in turn is a product of momentum invariants, $k \cdot \varepsilon$ factors, and $\varepsilon \cdot \varepsilon$ factors:

$$A_n \sim \sum \prod (j_1 j_2) \cdots k_{j_3} \cdot \varepsilon_{j_4} \cdots \varepsilon_{j_5} \cdot \varepsilon_{j_6} \cdots \tag{18}$$

One may think of the dot products as creating links between the different polarization vectors, or between polarization vectors and momenta. Each term in the squared matrix elements will contain an equal number of polarization vectors and their complex conjugates. Summing over helicities will produce a transverse projection operator; the on-shell gauge invariance of the partial amplitudes allow to choose any gauge we like for this transverse projection. As noted elsewhere [8], the ordinary helicities above will select the light-cone gauge form,

$$-g_{\mu\nu} + \frac{k_\mu q_\nu + q_\mu k_\nu}{q \cdot k} \tag{19}$$

where $q$ is the reference momentum for the given polarization vector. For our purposes here, it is however more convenient to think of the summation in Feynman gauge. Now, summing over the helicities can also be thought of as creating links, in this case between polarization vectors and their complex conjugates. Combining all these links into a chain, a term in the squared matrix element is now a product of factors, each of which has one of the two forms

$$\varepsilon_{j_1} \cdot \varepsilon_{j_2} \, \varepsilon^*_{j_2} \cdot \varepsilon^*_{j_3} \cdots \varepsilon_{j_4} \cdot \varepsilon_{j_1} \tag{20}$$

or

$$k \cdot \varepsilon_{j_2} \, \varepsilon^*_{j_2} \cdot \varepsilon^*_{j_3} \cdots \varepsilon_{j_4} \cdot k' \tag{21}$$

that is, the chain is either closed, or else it ends on two momenta. In Feynman gauge, each chain simply contracts (up to a sign) to a metric tensor. Thus in the case of the chain (21) ending on momenta, the factor simply produces the dot product $k \cdot k'$. So long as we impose only the constraints emerging from momentum conservation, and not additional constraints of the Asribekov-Byers-Yang type [12] (emerging from the dimension of space-time), these momentum invariants are the same in 4 and in $4 - \epsilon$ dimensions; this justifies keeping the external momenta in four dimensions. We are left only with the factors coming from closed chains (20). In these factors, we will eventually end up with a trace of the metric (up to a sign). The four-dimensional indices are taken care of by the usual four-dimensional helicities; and the reader may verify that the contribution to the trace coming from the $\epsilon$ dimensions is reproduced exactly by the prescription of equation (6). (The rule actually assigns indices to a dot product of two polarization vectors, rather than to the sum over helicities of a polarization vector times its complex conjugate; but once all indices are contracted into a trace, this shift is invisible.)

Siegel [13] introduced an alternative regularization scheme, called dimensional reduction, in which the number of gluon helicities is kept fixed at two (so as to match the number of fermion helicities in a supersymmetric theory). In using this scheme, one would of course not need the '$\epsilon$' helicities introduced above, but only the usual four-dimensional helicities. This suggests that the latter scheme would be more natural and simple to use within the framework of the spinor helicity method.

The version of dimensional regularization assumed above is that employed by Ellis and Sexton, in which all gluons, both internal and external, are continued to $4-\epsilon$ dimensions. In the original 't Hooft-Veltman scheme [14], only the gluons inside loops are dimensionally continued; the external (observed) particles are kept in 4 dimensions. Unitarity then requires the soft and collinear particles (that is, the *unobserved* ones) in the corresponding higher-point lower-loop diagrams also be continued to $4-\epsilon$ dimensions. (The higher-point lower-loop contributions cancel the infrared divergences when computing a physical matrix element.)

For the case of next-to-leading corrections to, say, an $n$-jet cross section, this in principle would mean that the lone soft or collinear gluon in the $(n+1)$-point tree contribution should also be treated in $4-\epsilon$ dimensions. However, by virtue of equations (7), all terms involving an odd number of epsilon helicities vanish; and thus in the case of next-to-leading corrections, this variant of dimensional regularization requires only the computation of the usual helicities. For computing corrections beyond the next-to-leading, however, the epsilon helicities would be required.

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